7.2. Trigonometric Integrals.

Identities.

$$\csc\theta = \frac{1}{\sin\theta}. \qquad \sec\theta = \frac{1}{\cos\theta}$$

$$\tan\theta = \frac{\sin\theta}{\cos\theta} \qquad \cot\theta = \frac{\cos\theta}{\sin\theta}$$

$$\sin^2\theta + \cos^2\theta = 1.$$

$$1 + \tan^2\theta = \sec^2\theta$$
$$1 + \cot^2\theta = \csc^2\theta.$$

Double-angle formulas:

$$\sin 2\theta = 2\sin\theta\cos\theta$$
$$\cos 2\theta = \cos^2\theta - \sin^2\theta$$
$$= 2\cos^2\theta - 1$$
$$= 1 - 2\sin^2\theta$$

Half-angle formulas.

$$\cos^2\theta = \frac{1+\cos 2\theta}{2}$$
$$\sin^2\theta = \frac{1-\cos 2\theta}{2}$$

Ex. $\int \sin^3 x\,dx$. $\quad \sin^3 x = \sin^2 x \sin x$

$$= (1-\cos^2 x)\sin x.$$

$u = \cos x$. $\quad du = -\sin x\,dx$

$$\int \sin^3 x\,dx = \int (1-\cos^2 x)\sin x\,dx$$
$$= -\int (1-u^2)\,du = -u + \frac{1}{3}u^3 + C$$
$$= -\cos x + \frac{1}{3}\cos^3 x + C.$$

Ex: $\int \sin^3 x \cos^2 x\,dx$

$$= \int \sin^2 x \sin x \cos^2 x\,dx$$
$$= \int (1-\cos^2 x) \qquad \sin x\,dx$$

$u = \cos x$
$du = -\sin x\,dx$

$$= -\int (1-u^2)u^2\,du$$
$$= -\int (u^2 - u^4)\,du$$
$$= -\frac{1}{3}u^3 + \frac{1}{5}u^5 + C = -\frac{1}{3}\cos^3 x + \frac{1}{5}\cos^5 x + C$$

Ex: $\int \cos^3 x \sin^2 x\,dx$

$$= \int \cos^2 x \cos x \sin^2 x\,dx$$
$$= \int (1-\sin^2 x)\sin^2 x \cos x\,dx$$

$u = \sin x$
$du = \cos x\,dx$

$$= \int (1-u^2)u^2\,du$$
$$= \int (u^2 - u^4)\,du$$
$$= \frac{1}{3}u^3 - \frac{1}{5}u^5 + C = \frac{1}{3}\sin^3 x - \frac{1}{5}\sin^5 x + C$$

Ex. $\int_0^{\pi} \cos^2 x \, dx$.

$$= \frac{1}{2}\int_0^{\pi} (1+\cos 2x)\,dx = \frac{1}{2}\left(x + \frac{1}{2}\sin 2x\right)\Big|_0^{\pi}$$
$$= \frac{\pi}{2}$$

$$\boxed{\begin{aligned} \sin x \cos y &= \frac{1}{2}\left[\sin(x-y) + \sin(x+y)\right] \\ \sin x \sin y &= \frac{1}{2}\left[\cos(x-y) - \cos(x+y)\right] \\ \cos x \cos y &= \frac{1}{2}\left[\cos(x-y) + \cos(x+y)\right] \end{aligned}}$$

Ex. $\int \cos x \cos 2x \, dx$

$$= \int \frac{1}{2}\left[\cos x + \cos 3x\right] dx$$
$$= \frac{1}{2}\int (\cos x + \cos 3x)\,dx$$
$$= \frac{1}{2}\sin x + \frac{1}{6}\sin 3x + C.$$

Evaluate $\int \tan^m x \sec^n x \, dx$ or $\int \cot^m x \csc^n x \, dx$.

Ex. $\int \tan^2 x \sec^4 x \, dx$

$$= \int \tan^2 x \sec^2 x \sec^2 x \, dx$$
$$= \int \tan^2 x (1+\tan^2 x) \sec^2 x \, dx \qquad u = \tan x, \quad du = \sec^2 x \, dx$$
$$= \int u^2 (1+u^2)\,du$$
$$= \frac{1}{3}u^3 + \frac{1}{5}u^5 + C = \frac{1}{3}\tan^3 x + \frac{1}{5}\tan^5 x + C$$

Ex: $\int \tan^3 x \sec^3 x\,dx$

$$= \int \tan^2 x \cdot \tan x \cdot \sec^2 x \cdot \sec x\,dx$$

$$= \int \tan^2 x \sec^2 x \cdot \tan x \sec x\,dx$$

$u = \sec x$
$du = \tan x \sec x\,dx$

$$= \int (\sec^2 x - 1) \sec^2 x \tan x \sec x\,dx$$

$$= \int (u^2 - 1) u^2\,du$$

$$= \frac{1}{5}u^5 - \frac{1}{3}u^3 + C$$

$$= \frac{1}{5}\sec^5 x - \frac{1}{3}\sec^3 x + C$$

Summary: ① even power of $\sec x$.

$\sec^2 x = 1 + \tan^2 x$

$$\sec^{2k} x = \sec^{2k-2} x \cdot \sec^2 x$$

② odd power of $\tan x$

$\tan^2 x = \sec^2 x - 1$.

$$\tan^{2k+1} x = \tan^{2k} x \cdot \tan x.$$