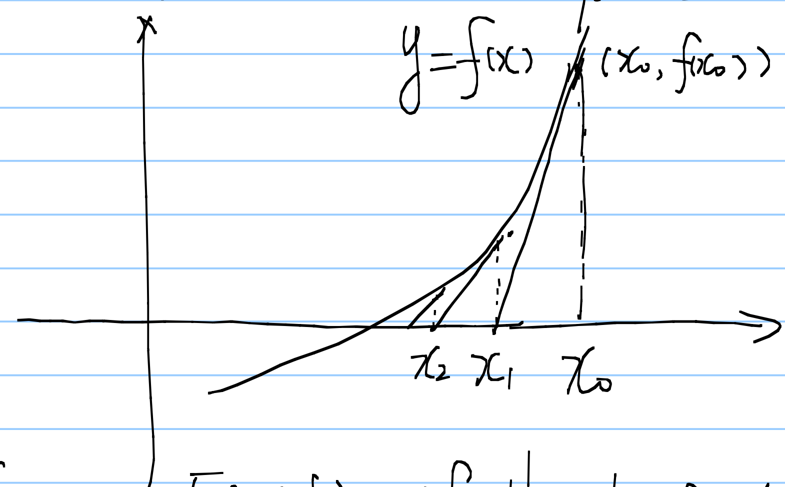
2.5 Newton's method

(Newton-Raphson method)

Idea:



$x_0 \to x_1$

Equation of the tangent line through $(x_0, f(x_0))$

$$y - f(x_0) = f'(x_0)(x - x_0)$$

x-intercept $\quad 0 - f(x_0) = f'(x_0)(x_1 - x_0)$

$$x_1 = x_0 - \frac{f(x_0)}{f'(x_0)}$$

Generally $x_n \rightarrow x_{n+1}$

$$\boxed{x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}} \leftarrow \text{Formula}$$

Stop criteria: $|f(x_{n+1})| < tol$ or $|x_{n+1} - x_n| < tol$

Ex 1 $f(x) = x^3 - 2x - 1$. $x_0 = 1.5$, $tol = 1 \times 10^{-2}$

Solution $f'(x) = 3x^2 - 2$

$x_0 \rightarrow x_1$ $\quad x_1 = x_0 - \frac{f(x_0)}{f'(x_0)} = 1.5 - \frac{(-0.625)}{4.75} = 1.631579$

$f(x_1) = 0.0801869$

$x_1 \rightarrow x_2$ $\quad x_2 = x_1 - \frac{f(x_1)}{f'(x_1)} = 1.618184$

$f(x_2) = 8.7589 \times 10^{-4}$ $\quad |f(x_2)| < 10^{-2}$. $\alpha \approx 1.618184$

Ex2: Develop an iterative procedure for $\sqrt[p]{q}$

Solution: Let $x = q^{\frac{1}{p}}$ $\quad f(x) = x^p - q = 0$

$$f'(x) = px^{p-1}$$

$x_0 \rightarrow x_1$

$$x_1 = x_0 - \frac{f(x_0)}{f'(x_0)} = x_0 - \frac{x_0^p - q}{px^{p-1}}$$

$$= \left(1 - \frac{1}{p}\right)x_0 + \frac{q}{p}x_0^{1-p}$$

$x_n - x_{n+1}$ $\quad x_{n+1} = \left(1 - \frac{1}{p}\right)x_n + \frac{q}{p}x_n^{1-p}$

Practice: $\sqrt{2} = ?$ $\quad x_0 = 1, \; p = q = 2$

$$x_1 = \left(1 - \frac{1}{2}\right)\cdot 1 + 1^{-1} = 1.5$$

$$x_2 = \left(1 - \frac{1}{2}\right) \cdot 1.5 + 1.5^{-1} = 0.75 + 0.66667$$

$$= 1.41667 = \frac{3}{4} + \frac{2}{3} = \frac{17}{12}$$

$$x_3 = \frac{1}{2} \cdot \frac{17}{12} + \frac{12}{17} \approx 1.414215686$$

$$\sqrt{2} \approx 1.41421356$$

Practice: $\sqrt[3]{2} = ?$