Runge-Kutta Methods

Taylor's Theorem for $f(x,y)$

$$f(x+h, y+k) = \sum_{i=0}^{n-1} \frac{1}{i!}\left(h\frac{\partial}{\partial x} + k\frac{\partial}{\partial y}\right)^i f(x,y) + \frac{1}{n!}\left(h\frac{\partial}{\partial x} + k\frac{\partial}{\partial y}\right)^n f(\bar{x}, \bar{y}) \qquad (1)$$

$$x \le \bar{x} \le x+h \quad , \quad y \le \bar{y} \le y+h$$

$$\left(h\frac{\partial}{\partial x} + k\frac{\partial}{\partial y}\right)^0 f(x,y) = f(x,y)$$

$$\left(h\frac{\partial}{\partial x} + k\frac{\partial}{\partial y}\right)^1 f(x,y) = h\frac{\partial}{\partial x} f(x,y) + k\frac{\partial}{\partial y} f(x,y) = hf_x(x,y) + kf_y(x,y)$$

$$\left(h\frac{\partial}{\partial x} + k\frac{\partial}{\partial y}\right)^2 f(x,y) = h^2 f_{xx}(x,y) + 2hk f_{xy}(x,y) + k^2 f_{yy}(x,y)$$

$$f(x+h, y+k) = f + (hf_x + kf_y) + \frac{1}{2!}(h^2 f_{xx} + 2hk f_{xy} + k^2 f_{yy})$$
$$+ \frac{1}{3!}(h^3 f_{xxx} + 3h^2 k f_{xxy} + 3hk^2 f_{xyy} + k^3 f_{yyy}) + \cdots \qquad (2)$$

Runge-Kutta Method of order 2

$$\begin{cases} \frac{dy}{dt} = f(t, y) \\ y(a) = y_0 \end{cases} \qquad \begin{cases} k_1 = hf(t_n, y_n) \\ k_2 = hf(t_n + \alpha h, y + \beta k_1) \end{cases}$$

$$y_{n+1} = y_n + w_1 k_1 + w_2 k_2$$

$$f(t_n + \alpha h, y_n + \beta hf) = f + \alpha h f_t + \beta h f f_y + \frac{1}{2}\left(\alpha h \frac{\partial}{\partial t} + \beta h f \frac{\partial}{\partial y}\right)^2 f(\bar{x}, \bar{y}) \qquad (3)$$

$$y(t_{n+1}) = y(t_n + h) = y(t_n) + (w_1 + w_2) hf + \alpha w_2 h^2 f_t + \beta w_2 h^2 f f_y + O(h^3) \qquad (4)$$

On the other hand

$$y(t_{n+1}) = y(t_n + h) = y(t_n) + h y'(t_n) + \frac{1}{2} h^2 y''(t_n) + O(h^3) \qquad (5)$$

$$y'(t_n) = f(t_n, y_n)$$

$$y''(t_n) = \frac{dy'}{dt} = \frac{d}{dt} f(t_n, y_n) = \frac{\partial f}{\partial t} + \frac{\partial f}{\partial y} \cdot \frac{dy}{dt} = f_t + f_x f \qquad (6)$$

$$y(t_{n+1}) = y(t_n) + hf + \frac{1}{2}h^2 f_t + \frac{1}{2}h^2 f f_x + O(h^3) \qquad (7)$$

(4) = (7) Agreement

$$w_1 + w_2 = 1, \quad \alpha w_2 = \frac{1}{2}, \quad \beta w_2 = \frac{1}{2}$$

One solution: $w_1 = w_2 = \frac{1}{2}$, $\alpha = \beta = 1$

$$y_{n+1} = y_n + \frac{1}{2}(k_1 + k_2)$$

$$\begin{cases} k_1 = hf(t_n, y_n) \\ k_2 = hf(t_n + h, y_n + k_1) \end{cases}$$

Modified Euler's Method

Another solution: $w_1 = 0$, $w_2 = 1$, $\alpha = \beta = \frac{1}{2}$

$$y_{n+1} = y_n + h k_2$$

$$\begin{cases} k_1 = hf(t_n, y_n) \\ k_2 = hf(t_n + \frac{h}{2}, y_n + \frac{1}{2}k_1) \end{cases}$$

<u>Ex</u>

$$\begin{cases} \frac{dy}{dt} = 2 + (y - t - 1)^2 \\ y(1) = 2 \end{cases}$$

Exact solution: $y(t) = 1 + t + \tan(t-1)$

$h = 0.01$, $n = 100$, $t_0 = 1$, $\ldots$ $t_n = 2$

Runge-Kutta Method of Order 4

$$y_{n+1} = y_n + \frac{1}{6}(k_1 + 2k_2 + 2k_3 + k_4)$$

$$\begin{cases} k_1 = hf(t_n, y_n) \\ k_2 = hf(t_n + \frac{1}{2}h, \ y_n + \frac{1}{2}k_1) \\ k_3 = hf(t_n + \frac{1}{2}h \ y_n + \frac{1}{2}k_2) \\ k_4 = hf(t_n + h, \ y_n + k_3) \end{cases}$$